

Dirichlet problem for the constant mean curvature equation and CMC foliation in the extended Schwarzschild spacetime

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Abstract

We prove the existence and uniqueness of the Dirichlet problem for space-like, spherically symmetric, constant mean curvature equation with symmetric boundary data in the extended Schwarzschild spacetime. As an application, we completely solve the CMC foliation conjecture which is posted by Malec and Ó Murchadha in [11].

1 Introduction

Spacelike constant mean curvature (CMC) hypersurfaces in spacetimes have been considered as important objects in general relativity. This is because constant mean curvature hypersurfaces are used in the analysis on Einstein constraint equations [10, 6] and the gauge condition in the Cauchy problem of the Einstein equations [2, 5]. In addition, York suggested the concept of the CMC foliation and the CMC time function on relativistic cosmology [13], so CMC hypersurfaces could characterize the global structure of cosmological spacetimes.

The Schwarzschild spacetime is the simplest model of a universe containing a star. Its metric is a solution of the vacuum Einstein equation, and it is spherically symmetric and asymptotically flat. The maximal analytic extension of the Schwarzschild spacetime was obtained by Kruskal in 1960 and nowadays it is called the Kruskal extension.

Some results of spacelike, spherically symmetric constant mean curvature hypersurfaces (SS-CMC) in the Schwarzschild spacetime and in the Kruskal extension can be found in Brill, Cavallo, and Isenberg's paper [4], and in Malec and Ó Murchadha's papers [11, 12]. In papers [8, 9], the authors considered the initial value problem of the SS-CMC equation in the Schwarzschild spacetime. These SS-CMC solutions are completely solved and characterized by two constants of integration. Furthermore,

the authors discussed the behaviors of SS-CMC hypersurfaces near the coordinate singularities, and then gave the correspondences between SS-CMC hypersurfaces in the Schwarzschild spacetime and SS-CMC hypersurfaces in the Kruskal extension. Thus, the initial value problem of the SS-CMC equation in the Kruskal extension is solved as well.

In this paper, we consider the Dirichlet problem for the SS-CMC equation in the Kruskal extension with T -axisymmetric boundary data. The result is comprehensively stated below:

Theorem. *The Dirichlet problem for the SS-CMC equation and symmetric boundary data in the Kruskal extension is solvable and the solution is unique.*

The precise setting of the SS-CMC Dirichlet problem is in section 4.1. For the existence part, we use the shooting method since the boundary value problem can be reduced to the initial value problem, which is solved in papers [8, 9]. The uniqueness part is much difficult. We introduce the Lorentzian distance function with respect to some spacelike hypersurface. The maximum point of the Laplacian of the Lorentzian distance function restrict on another spacelike hypersurface has a good estimate related to the mean curvatures of both spacelike hypersurfaces. Thus, if the solution of the SS-CMC Dirichlet problem is not unique, we can apply the Lorentzian distance function estimate to two of SS-CMC solutions and then get a contradiction.

As an application, we prove that the T -axisymmetric SS-CMC foliation (we use TSS-CMC foliation for short) property conjectured by Malec and Ó Murchadha in [11] is true. In paper [11], they constructed a particular family of TSS-CMC hypersurfaces and claimed this family foliates the Kruskal extension. In paper [9], the authors reformulated the TSS-CMC foliation conjecture and proved some partial results. In this paper, we explain the existence of the Dirichlet problem is equivalent to the family of TSS-CMC hypersurfaces cover the Kruskal extension, and the uniqueness of the Dirichlet problem is equivalent to any two TSS-CMC hypersurfaces are disjoint, and hence the TSS-CMC foliation conjecture is proved.

The organization of this paper is as follows. We first give a brief summary of the Schwarzschild spacetime and Kruskal extension in section 2.1, and then state the main results of the initial value problem of the SS-CMC equation in the Kruskal extension in section 2.2. The Lorentzian distance function and its properties are discussed in section 3. These results are used to solve the SS-CMC Dirichlet problem in section 4. In section 5, we will prove the TSS-CMC foliation conjecture.

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2 Preliminary

2.1 The Schwarzschild spacetime and the Kruskal extension

The Schwarzschild spacetime is a 4-dimensional time-oriented Lorentzian manifold with metric

$$ds^2 = - \left(1 - \frac{2M}{r} \right) dt^2 + \frac{1}{\left(1 - \frac{2M}{r} \right)} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2.$$

After coordinates change, the Schwarzschild metric can be written as

$$\begin{aligned} ds^2 &= \frac{16M^2 e^{-\frac{r}{2M}}}{r} (-dT^2 + dX^2) + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \\ &= \frac{16M^2 e^{-\frac{r}{2M}}}{r} dU dV + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2, \end{aligned} \quad (1)$$

where

$$\begin{cases} (r - 2M) e^{\frac{r}{2M}} = X^2 - T^2 = VU \\ \frac{t}{2M} = \ln \left| \frac{X + T}{X - T} \right| = \ln \left| \frac{V}{U} \right|. \end{cases} \quad (2)$$

From (1), we know that $r = 2M$ is only a coordinate singularity. The Schwarzschild spacetime has a maximal analytic extension, called the Kruskal extension. It is the union of regions I, II, I', and II', where regions I and II correspond to the exterior and interior of one Schwarzschild spacetime, respectively, and regions I' and II' correspond to the exterior and interior of another Schwarzschild spacetime. Figure 1 points out their correspondences and coordinate systems (X, T) or (U, V) .

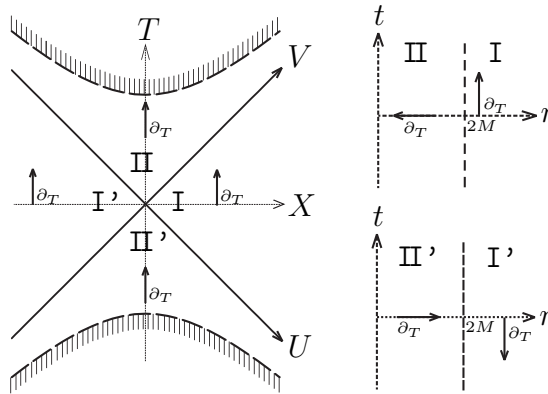


Figure 1: The Kruskal extension of Schwarzschild spacetimes.

Sometimes we will use another null coordinates (u, v) by

$$u = t - (r + 2M \ln |r - 2M|) \quad \text{and} \quad v = t + (r + 2M \ln |r - 2M|), \quad (3)$$

and relations between (U, V) and (u, v) are given by

	region I	region II	region I'	region II'
U	$e^{-\frac{u}{4M}}$	$-e^{-\frac{u}{4M}}$	$-e^{-\frac{u}{4M}}$	$e^{-\frac{u}{4M}}$
V	$e^{\frac{v}{4M}}$	$e^{\frac{v}{4M}}$	$-e^{\frac{v}{4M}}$	$-e^{\frac{v}{4M}}$

In this article, we will take ∂_T as a future-directed timelike vector field in the Kruskal extension. Note that ∂_T in two Schwarzschild spacetimes has different directions and it is pointed out in Figure 1. Once ∂_T is chosen, for a spacelike hypersurface Σ , we will choose the normal vector \vec{n} of Σ as future-directed in the Kruskal extension, and the mean curvature H of Σ is defined by $H = \frac{1}{3}g^{ij}\langle\nabla_{e_i}\vec{n}, e_j\rangle$, where $\{e_i\}_{i=1}^3$ is a basis on Σ .

2.2 SS-CMC hypersurfaces in the Kruskal extension

In this subsection, we will summarize the results in papers [8] and [9] about spherically symmetric, spacelike constant mean curvature (SS-CMC) hypersurfaces in the Schwarzschild spacetime and Kruskal extension. These formulae and arguments will be used to deal with the SS-CMC Dirichlet problem in section 4. We refer to papers [8, 9] for more details.

In papers [8, 9], we considered the initial value problem of the SS-CMC equation in the Schwarzschild spacetime and Kruskal extension. We first studied SS-CMC hypersurfaces in the standard Schwarzschild coordinates. In most of cases, an SS-CMC hypersurface Σ can be locally written as a graph of $(t = f(r), r, \theta, \phi)$ in either exterior or interior of the Schwarzschild spacetime. Direct computation gives the SS-CMC equation, which is a second-order ordinary differential equation:

$$f'' + \left(\left(\frac{1}{h} - (f')^2 h \right) \left(\frac{2h}{r} + \frac{h'}{2} \right) + \frac{h'}{h} \right) f' \pm 3H \left(\frac{1}{h} - (f')^2 h \right)^{\frac{3}{2}} = 0, \quad (4)$$

where $h(r) = 1 - \frac{2M}{r}$, $H \in \mathbb{R}$ is the constant mean curvature, and the spacelike condition is equivalent to $\frac{1}{h} - (f')^2 h > 0$. The choice of \pm signs in the equation (4) depends on different regions that cause different directions of the normal vector and different pieces of an SS-CMC hypersurface. In each region and where $f(r)$ is defined, the solution $f(r)$ is uniquely determined by two constants of integration, denoted by c and \bar{c} , where c controls the slope of the function of $f(r)$ and \bar{c} represents the t -direction translation. Thus, if we require that an SS-CMC hypersurface $\Sigma_{H,c,\bar{c}}$ passes through a point (t_0, r_0) , then \bar{c} is determined.

All solutions of the SS-CMC equation can be expressed in the integration form, which are completely discussed in [8, 9]. Because of symmetry, only parts of expressions of $\Sigma_{H,c,\bar{c}}$ are needed in this paper, and we summarize these formulae below.

(A) For $\Sigma_{H,c,\bar{c}}$ in the Schwarzschild exterior (maps to region I), then

$$f(r; H, c, \bar{c}) = \int_{r_1}^r \frac{l(x; H, c)}{h(x)\sqrt{1+l^2(x; H, c)}} dx + \bar{c},$$

where $l(r; H, c) = \frac{1}{\sqrt{h(r)}} \left(Hr + \frac{c}{r^2} \right)$, and $r_1 \in (2M, \infty)$ is a fixed number.

If $c < -8M^3H$, then $f(r) \rightarrow \infty$ as $r \rightarrow 2M^+$ with asymptotic behavior $f'(r) = \frac{1}{h(r)} + \bar{f}'(r)$, where

$$\bar{f}'(r) = \frac{r^4}{(Hr^3 + c)^2 + r^3(r - 2M) - (Hr^3 + c)\sqrt{(Hr^3 + c)^2 + r^3(r - 2M)}}. \quad (5)$$

If $c > -8M^3H$, then $f(r) \rightarrow -\infty$ as $r \rightarrow 2M^+$ with asymptotic behavior $f'(r) = -\frac{1}{h(r)} + \tilde{f}'(r)$, where

$$\tilde{f}'(r) = \frac{-r^4}{(Hr^3 + c)^2 + r^3(r - 2M) + (Hr^3 + c)\sqrt{(Hr^3 + c)^2 + r^3(r - 2M)}}. \quad (6)$$

If $c = -8M^3H$, then $f(r)$ tends to some finite value as $r \rightarrow 2M^+$.

(B) For $\Sigma_{H,c,\bar{c}}$ in the Schwarzschild interior (region II), then

$$f(r; H, c, \bar{c}) = \int_{r_2}^r \frac{l(x; H, c)}{\mp h(x)\sqrt{l^2(x; H, c) - 1}} dx + \bar{c},$$

where $l(r; H, c) = \frac{1}{\sqrt{-h(r)}} \left(-Hr - \frac{c}{r^2} \right) > 1$, and r_2 is in the domain of $f(r)$.

If $r \rightarrow 2M^-$ and $f'(r) > 0$, then $f(r) \rightarrow \infty$ with asymptotic behavior $f'(r) = -\frac{1}{h(r)} + \bar{f}'(r)$, where $\bar{f}'(r)$ is (5).

(C) For $\Sigma_{H,c,\bar{c}}$ in the Schwarzschild interior (region II'), then

$$f(r; H, c, \bar{c}) = \int_{r_4}^r \frac{l(x; H, c)}{\pm h(x)\sqrt{l^2(x; H, c) - 1}} dx + \bar{c},$$

where $l(r; H, c) = \frac{1}{\sqrt{-h(r)}} \left(Hr + \frac{c}{r^2} \right) > 1$, and r_4 is in the domain of $f(r)$.

If $r \rightarrow 2M^-$ and $f'(r) < 0$, then $f(r) \rightarrow -\infty$ with asymptotic behavior $f'(r) = \frac{1}{h(r)} + \tilde{f}'(r)$, where $\tilde{f}'(r)$ is (6).

(D) If $c < -8M^3H$, we can find \bar{c} such that hypersurfaces in region I and II can smoothly glue together at $U = 0$ ($r = 2M$ and $t = \infty$).

If $c > -8M^3H$, we can find \bar{c} such that hypersurfaces in region I and II' can smoothly glue together at $V = 0$ ($r = 2M$ and $t = -\infty$).

If $c = -8M^3H$, we can find \bar{c} such that hypersurfaces in region I and I' can smoothly glue together at $(U, V) = (0)$ ($r = 2M$ and t is finite).

Only constant slices $r = r_0, r_0 \in (0, 2M)$ are SS-CMC hypersurfaces which can *not* be written as a graph of $t = f(r)$. These hypersurfaces are called *cylindrical hypersurfaces* and they have constant mean curvature

$$H(r_0) = \frac{2r_0 - 3M}{3\sqrt{r_0^3(2M - r_0)}} \text{ in region } \Pi, \text{ or } H(r_0) = \frac{3M - 2r_0}{3\sqrt{r_0^3(2M - r_0)}} \text{ in region } \Pi'.$$

The behavior of an SS-CMC hypersurface $\Sigma_{H,c,\bar{c}}$ in the Kruskal extension highly depends on the constant of integration c . Precisely, fixed $H \in \mathbb{R}$, define two functions $k_H(r)$ and $\tilde{k}_H(r)$ on $(0, 2M]$ by

$$k_H(r) = -Hr^3 - r^{\frac{3}{2}}(2M - r)^{\frac{1}{2}} \quad \text{and} \quad \tilde{k}_H(r) = -Hr^3 + r^{\frac{3}{2}}(2M - r)^{\frac{1}{2}}.$$

These two functions come from the condition $l > 1$ in the Schwarzschild interior and their graphs are plotted in Figure 2 (a)¹.

Denote (r_H, c_H) by the minimum point of $k_H(r)$ and (R_H, C_H) by the maximum point of $\tilde{k}_H(r)$. We use $k_H^+(r)$, $k_H^-(r)$, $\tilde{k}_H^+(r)$, and $\tilde{k}_H^-(r)$ to represent their increasing or decreasing part. Notice that both $r = r_H$ and $r = R_H$ are cylindrical SS-CMC hypersurfaces with constant mean curvature H in region Π and Π' , respectively. They correspond to hyperbolas in the Kruskal extension, and thus they divide the Kruskal extension into three regions, called the top region, the middle region, and the bottom region. See Figure 2 (b).

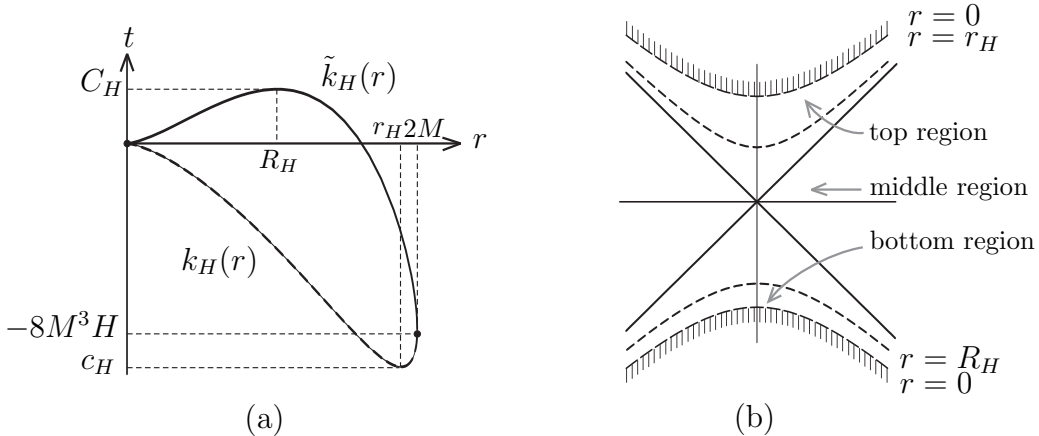


Figure 2: (a) Graphs of $k_H(r)$ and $\tilde{k}_H(r)$. (b) SS-CMC hypersurfaces $r = r_H$ and $r = R_H$, and they divide the Kruskal extension into three regions.

Now we summarize results of the initial value problem of the SS-CMC equation in the Kruskal extension. Given $H \in \mathbb{R}$, for (T_0, X_0) in the middle region as Figure 3, and for $c \in \mathbb{R}$, we can find an SS-CMC hypersurface $\Sigma_{H,c}$ passing through (T_0, X_0) .

¹ All figures in this paper are presented the case $H > 0$. Cases $H = 0$ or $H \leq 0$ are similarly discussed.

There are three values $c = c_H, c = -8M^3H$, and $c = C_H$ such that the value c in different interval determines different behavior of the SS-CMC hypersurface $\Sigma_{H,c}$, which is also illustrated in Figure 3. Remark that for $c \in (c_H, C_H)$, the solution of $c = k_H^+(r)$ or $c = \tilde{k}_H^-(r)$, which is denoted by $r = r_{H,c}$, is called “throat” of the SS-CMC hypersurface $\Sigma_{H,c}$. The throat indicates that $\Sigma_{H,c}$ is not defined in $r \in (0, r_{H,c})$ in the Schwarzschild interior. Notice that each $\Sigma_{H,c}$ is diffeomorphic to $I \times \mathbb{S}^2$, where $I \subset \mathbb{R}$, so the throat $r = r_{H,c}$ is the smallest radius of the Schwarzschild coordinates sphere in $\Sigma_{H,c}$. Furthermore, every SS-CMC hypersurface $\Sigma_{H,c,\bar{c}}, c \in (c_H, C_H)$ is symmetric with respect to its throat in the Schwarzschild coordinates. See the discussion in [8, Proposition 2.6].

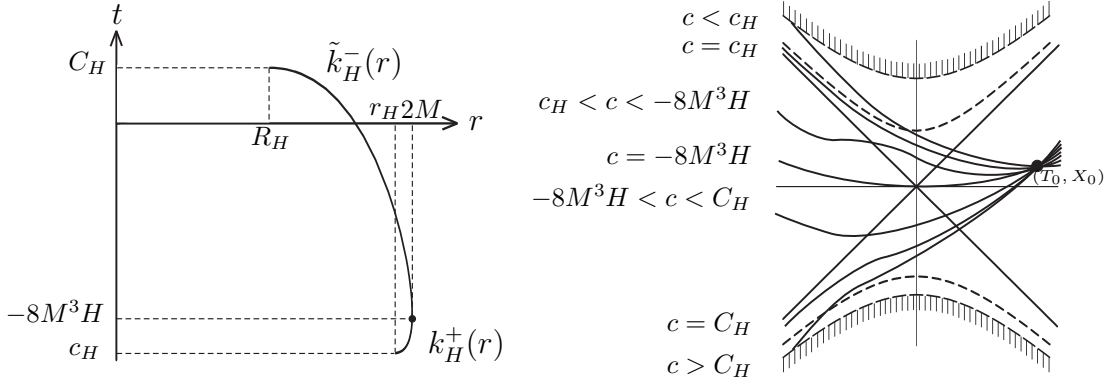


Figure 3: SS-CMC initial value problem when (T_0, X_0) is in the middle region.

For (T_0, X_0) in the top region (or the bottom region), as Figure 4, denote (t_0, r_0) by its Schwarzschild coordinates, then for $c \leq k_H^-(r_0)$ (or $c \geq \tilde{k}_H^+(r_0)$), we can find an SS-CMC hypersurface $\Sigma_{H,c}$ with $f'(r_0) < 0$ (or $f'(r_0) > 0$) in the Schwarzschild interior passing through (T_0, X_0) . There is also an SS-CMC hypersurface $\Sigma_{H,c}$ with $f'(r_0) > 0$ (or $f'(r_0) < 0$) in the Schwarzschild interior passing through (T_0, X_0) , but it is not illustrated in Figure 4.

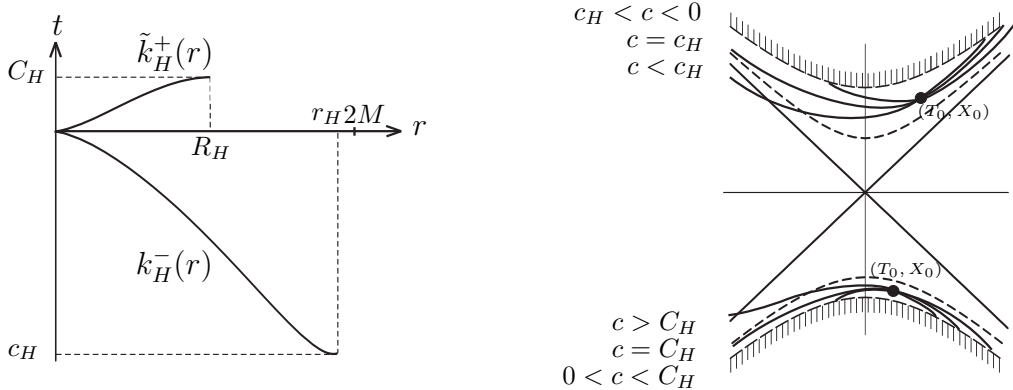


Figure 4: SS-CMC initial value problem when (T_0, X_0) is in the top or bottom region.

3 The Lorentzian distance function

We will introduce the Lorentzian distance function from a fixed achronal spacelike hypersurface in this section, and then derive a formula of the Laplacian of the Lorentzian distance function for further analysis on the Dirichlet problem of the SS-CMC equation in section 4. Here we focus on the spacetime case. Remark that results in this section hold in general n -dimensional Lorentzian manifold. Most of results in this section can also be found in papers [1, 7] and in their references.

Let N^3 be an achronal spacelike hypersurface in a spacetime $(L^4, \langle \cdot, \cdot \rangle)$. Given $p, q \in L$, define the *Lorentzian distance function* $d(p, q) : L \times L \rightarrow [0, \infty]$ as follows:

- (a) If $q \in J^+(p)$, then $d(p, q)$ is the supremum of the Lorentzian lengths of all the future-directed causal curve from p to q .
- (b) If $q \notin J^+(p)$, then $d(p, q) = 0$.

Remark that the Lorentzian distance function is *not* symmetric, that is, $d(p, q) \neq d(q, p)$. For $q \in L$, we can further define the *Lorentzian distance function* $d_N : L \rightarrow [0, \infty]$ *with respect to* N as $d_N(q) = \sup_{p \in N} d(p, q)$.

In general, the Lorentzian distance function d_N is *not* smooth for every point in L , but it is smooth in a sufficiently near chronological future of N . To state this result, let ν be the future-directed unit timelike normal vector field of N . We define a function $s_N : N \rightarrow [0, \infty]$ by $s_N(p) = \sup\{t \geq 0 : d_N(\gamma(t)) = t\}$, where $\gamma(t)$ is the future inextensible geodesic such that $\gamma(0) = p$ and $\gamma'(0) = \nu$. Denote $\tilde{\mathcal{I}}^+(N) = \{t\nu : \text{for all } p \in N \text{ and } 0 < t < s_N(p)\}$, and consider the set

$$\mathcal{I}^+(N) = \exp_N(\text{int}(\tilde{\mathcal{I}}^+(N))) \subset I^+(N),$$

where \exp_N is the exponential map with respect to N .

Lemma 1. [1, Lemma 5.1], [7, Proposition 3.6] *If $\mathcal{I}^+(N) \neq \emptyset$, then d_N is smooth on $\mathcal{I}^+(N)$ and its gradient $\overline{\nabla} d_N$ is a past-directed unit timelike vector field on $\mathcal{I}^+(N)$. That is, $\overline{\nabla} d_N = -\nu$.*

The analysis on the Lorentzian distance function is related to the Jacobi field, and we summarize the results here. Denote $\overline{\nabla}$ the connection of the spacetime L compatible with the metric $\langle \cdot, \cdot \rangle$, and ∇ the connection of N induced from L . Define A_N as the *shape operator* of N with respect to L by $A_N(X) = -\nabla_X \nu$ for future-directed unit timelike normal vector field ν of N and for tangent vector field X of N . Given $p \in N$, let $\gamma(t)$ be the future-directed unit timelike geodesic with $\gamma(0) = p$

and $\gamma'(0) = \nu$. We say a vector field $J(t)$ on the geodesic $\gamma(t)$ is a *normal N -Jacobi vector field* if $J''(t) + R(J(t), \gamma'(t))\gamma'(t) = 0$, $J(0) \in T_p N$, and $J'(0) = -A_N(J(0))$.

When we write a normal N -Jacobi field $J(t)$ in term of a parallel orthonormal frame, by the existence and uniqueness of the ordinary differential system, for $v \in T_p N$, there exists a unique N -Jacobi field $J(t)$ along $\gamma(t)$ satisfies $J(0) = v$, so the set of all normal N -Jacobi fields $J(t)$ along $\gamma(t)$ is of dimension n .

The point $q \in L$ is called a *focal point* of N if there exists a geodesic $\gamma : [0, l] \rightarrow L$ with $\gamma(0) = p \in N$, $\gamma'(0) \in (T_p N)^\perp$, $\gamma(l) = q \in L$, and a nonzero normal N -Jacobi field $J(t)$ along $\gamma(t)$ such that $J(l) = 0$.

Similar to the Riemannian case, we can compute the Hessian of the Lorentzian distance function $\overline{\text{Hess}}(d_N)$ in terms of the normal N -Jacobi vector field J and the curvature tensor R of the spacetime L .

Proposition 2. [7, Proposition 3.7] *Let N be an achronal spacelike hypersurface in a spacetime L . If $\mathcal{I}^+(N) \neq \emptyset$, $q \in \mathcal{I}^+(N)$ and $Y \in T_q L$ is a nonzero vector orthogonal to $\overline{\nabla} d_N(q)$, then*

$$\begin{aligned} \overline{\text{Hess}}(d_N(q))(Y, Y) &= - \int_0^l (\langle J'(t), J'(t) \rangle - \langle R(J(t), \gamma'(t))\gamma'(t), J(t) \rangle) dt \\ &\quad + \langle A_N(J(0)), J(0) \rangle, \end{aligned}$$

where $\gamma(t) : [0, l] \rightarrow \mathcal{I}^+(N)$, $\gamma(0) = p$, $\gamma'(0) \in (T_p N)^\perp$ and $\gamma(l) = q$ is the future-directed unit timelike geodesic, $J(t)$ is the unique normal N -Jacobi field along $\gamma(t)$ with $J(l) = Y$, and $J'(0) = -A_N(J(0))$.

Proof. Since $\gamma(t)$ is realized by the Lorentzian distance function, we have $\gamma'(t) = -\overline{\nabla} d_N(\gamma(t))$ for all $t \in (0, l]$. If $J(t)$ is the unique normal N -Jacobi field along $\gamma(t)$, then we have

$$\begin{aligned} \overline{\text{Hess}}(d_N)(Y, Y) &= \overline{\text{Hess}}(d_N)(J(l), J(l)) \\ &= \langle J(l), \overline{\nabla}_{J(l)} \overline{\nabla} d_N \rangle = -\langle J(l), \overline{\nabla}_{\gamma'(l)} J(l) \rangle = -\langle J(l), J'(l) \rangle \\ &= - \int_0^l \frac{d}{dt} \langle J(t), J'(t) \rangle dt - \langle J(0), J'(0) \rangle \\ &= - \int_0^l (\langle J'(t), J'(t) \rangle - \langle R(J(t), \gamma'(t))\gamma'(t), J(t) \rangle) dt + \langle A_N(J(0)), J(0) \rangle. \end{aligned}$$

□

From the formula of $\overline{\text{Hess}}(d_N)$, it is natural to define the *index form* of the geodesic $\gamma(t)$ with respect to N :

$$I_N(V, W) = - \int_0^l (\langle V', W' \rangle - \langle R(V, \gamma')\gamma', W \rangle) dt + \langle A_N(V(0)), W(0) \rangle.$$

where V and W are vector fields along and orthogonal to γ .

The following theorem shows that the normal N -Jacobi vector field maximizes the index form I_N .

Theorem 3. [1, Theorem 5.4] *Let N be an achronal spacelike hypersurface in a spacetime L . Given $p \in N$, let $\gamma(t) : [0, l] \rightarrow \mathcal{I}^+(N)$, $\gamma(0) = p$ and $\gamma'(0) \in T_p N$ be the future-directed unit timelike geodesic. Suppose there are no focal points to N along γ . Let J be a normal N -Jacobi field along γ . Then for every vector field X along and orthogonal to γ such that $X(l) = J(l)$, it holds that*

$$I_N(J, J) \geq I_N(X, X),$$

with equality if and only if $J = X$.

Proof. Let $\{e_i(t)\}_{i=1}^3$ be a parallel orthonormal frame along γ and orthogonal to $\gamma'(t)$, then we can write

$$J(t) = \sum_{i=1}^3 J_i(t) e_i(t), \quad \text{and} \quad X(t) = \sum_{i=1}^3 X_i(t) J_i(t) e_i(t), \quad X_i(l) = 1.$$

It implies $X'(t) = \sum_{i=1}^3 X'_i(t) J_i(t) e_i(t) + X_i(t) J'_i(t) e_i(t) = A(t) + B(t)$, and

$$I_N(X, X) = - \int_0^l \langle A, A \rangle + 2\langle A, B \rangle + \langle B, B \rangle - \langle R(X, \gamma') \gamma', X \rangle dt + \langle A_N(X(0)), X(0) \rangle.$$

Since

$$\begin{aligned} \langle X(t), B(t) \rangle' &= \langle X'(t), B(t) \rangle + \langle X(t), B'(t) \rangle \\ &= \langle A + B, B \rangle + \left\langle \sum_{i=1}^3 X_i J_i e_i, \sum_{j=1}^3 X'_j J'_j e_j + X_j J''_j e_j \right\rangle \\ &= 2\langle A, B \rangle + \langle B, B \rangle + \sum_{i,j=1}^3 \langle X_i J_i e_i, X_j J''_j e_j \rangle, \end{aligned}$$

we have

$$\begin{aligned} I_N(X, X) &= - \int_0^l \langle A, A \rangle + \langle X(t), B(t) \rangle' - \sum_{j=1}^3 \langle X, X_j J''_j e_j \rangle - \langle R(X, \gamma') \gamma', X \rangle dt \\ &\quad + \langle A_N(X(0)), X(0) \rangle \\ &= - \int_0^l \langle A, A \rangle dt - \langle X(l), B(l) \rangle + \langle X(0), B(0) \rangle + \langle A_N(X(0)), X(0) \rangle \\ &= - \int_0^l \langle A, A \rangle dt + I_N(J, J) + \langle A_N(X(0)) + B(0), X(0) \rangle \leq I_N(J, J). \end{aligned}$$

□

Next, we will get the estimate an Laplacian of the Lorentzian distance function, where the spacetime satisfies the timelike convergence condition. For $q \in L$, we say $p \in N$ is the *orthogonal projection of q on N* if there is a timelike geodesic $\gamma : [0, l] \rightarrow L$ such that $\gamma(0) = p, \gamma(l) = q$ and $\gamma'(0) \in (T_p N)^\perp$.

Lemma 4. [1, Lemma 5.7] *Let L be a spacetime such that $\text{Ric}(Z, Z) \geq 0$ for every unit timelike vector Z , and let N be an achronal spacelike hypersurface such that $\mathcal{I}^+(N) \neq \emptyset$ and let $q \in \mathcal{I}^+(N)$. Then*

$$\overline{\Delta}d_N(q) \geq -3H_N(p),$$

where $\overline{\Delta}$ stands for the Lorentzian Laplacian operator on L , H_N is the mean curvature of the hypersurface N with respect to ν , and p is the orthogonal projection of q on N .

Proof. Let $\gamma : [0, l] \rightarrow L$ be the normal future-directed unit timelike geodesic with $\gamma(0) = p$ and $\gamma'(0) \in (T_p N)^\perp$. Let $\{e_i\}_{i=1}^3$ be orthonormal vectors in $T_q L$ orthogonal to $\gamma'(l) = -\overline{\nabla}d_N(q)$ and $J_i(t)$ be a normal N -Jacobi vector field along γ with $J_i(l) = e_i$ and $J'_i(0) = -A_N J_i(0)$. For every $\{X_i(t)\}_{i=1}^4$ orthonormal frame of parallel vector fields along γ such that $X_i(l) = e_i$ for $i = 1, \dots, 3$ and $X_4(t) = \gamma'(t)$, we have

$$\begin{aligned} \overline{\Delta}d_N(q) &= \sum_{i=1}^3 \overline{\text{Hess}}(d_N(q))(e_i, e_i) = \sum_{i=1}^3 \overline{\text{Hess}}(d_N(q))(J_i(l), J_i(l)) \\ &= \sum_{i=1}^3 I_N(J_i(l), J_i(l)) \geq \sum_{i=1}^3 I_N(X_i(l), X_i(l)) \\ &= - \int_0^l \left(\sum_{i=1}^3 \langle X'_i, X'_i \rangle - \sum_{i=1}^3 \langle R(X_i, \gamma')\gamma', X_i \rangle \right) dt + \sum_{i=1}^3 \langle A_N X_i(0), X_i(0) \rangle \\ &= \int_0^l \text{Ric}(\gamma'(t), \gamma'(t)) dt - 3H_N(p) \geq -3H_N(p). \end{aligned}$$

□

If we restrict the Lorentzian distance function $d_N : L \rightarrow [0, +\infty]$ on a spacelike hypersurface Σ , which denotes $d_N|_\Sigma : \Sigma \rightarrow [0, +\infty]$, we will find relations between $\overline{\text{Hess}}(d_N)$ and $\text{Hess}(d_N|_\Sigma)$. Since $\overline{\nabla}d_N = (\overline{\nabla}d_N)^\top + (\overline{\nabla}d_N)^\perp = \nabla(d_N|_\Sigma) - \langle \overline{\nabla}d_N, \nu \rangle \nu$ along Σ , where $\nabla(d_N|_\Sigma)$ is the gradient of $d_N|_\Sigma$ on Σ , we have

$$\begin{aligned} \langle \overline{\nabla}d_N, \overline{\nabla}d_N \rangle &= \langle \nabla(d_N|_\Sigma) - \langle \overline{\nabla}d_N, \nu \rangle \nu, \nabla(d_N|_\Sigma) - \langle \overline{\nabla}d_N, \nu \rangle \nu \rangle \\ &= |\nabla(d_N|_\Sigma)|^2 - |\langle \overline{\nabla}d_N, \nu \rangle|^2 = -1. \end{aligned}$$

Since $\langle \bar{\nabla} d_N, \nu \rangle > 0$, we have $\langle \bar{\nabla} d_N, \nu \rangle = \sqrt{1 + |\nabla(d_N|_\Sigma)|^2}$ and hence $\bar{\nabla} d_N = \nabla(d_N|_\Sigma) - \sqrt{1 + |\nabla(d_N|_\Sigma)|^2} \nu$. Next, for any vector field $X \in T\Sigma$, we have

$$\begin{aligned} \bar{\nabla}_X \bar{\nabla} d_N &= (\bar{\nabla}_X \bar{\nabla} d_N)^\top + (\bar{\nabla}_X \bar{\nabla} d_N)^\perp \\ &= \left(\bar{\nabla}_X \left(\nabla(d_N|_\Sigma) - \sqrt{1 + |\nabla(d_N|_\Sigma)|^2} \nu \right) \right)^\top \\ &\quad + \left(\bar{\nabla}_X \left(\nabla(d_N|_\Sigma) - \sqrt{1 + |\nabla(d_N|_\Sigma)|^2} \nu \right) \right)^\perp \\ &= \nabla_X \nabla(d_N|_\Sigma) + \sqrt{1 + |\nabla(d_N|_\Sigma)|^2} A_\Sigma(X) \\ &\quad - \langle A_\Sigma(X), \nabla(d_N|_\Sigma) \rangle \nu - X \left(\sqrt{1 + |\nabla(d_N|_\Sigma)|^2} \right) \nu. \end{aligned}$$

Thus,

$$\begin{aligned} \text{Hess}(d_N|_\Sigma)(X, X) &= \langle \nabla_X \nabla(d_N|_\Sigma), X \rangle \\ &= \overline{\text{Hess}}(d_N)(X, X) - \sqrt{1 + |\nabla(d_N|_\Sigma)|^2} \langle A_\Sigma(X), X \rangle. \end{aligned}$$

After taking trace, we get for all $q \in \Sigma$,

$$\Delta_\Sigma(d_N|_\Sigma)(q) = \overline{\Delta} d_N(q) + \overline{\text{Hess}}(d_N(q))(\nu, \nu) + 3H_\Sigma(q) \sqrt{1 + |\nabla(d_N|_\Sigma)(q)|^2}. \quad (7)$$

Remark the the above discussion can also be found in [1, page 5091]. The result of Lemma 4 and the equation (7) will imply an important inequality of $\Delta_\Sigma(d_N|_\Sigma)$.

Proposition 5. [1, Proposition 5.9] *Let L be a spacetime such that $\text{Ric}(Z, Z) \geq 0$ for every unit timelike vector Z , and let N be an achronal spacelike hypersurface with $\mathcal{I}^+(N) \neq \emptyset$. Suppose that a spacelike hypersurface Σ satisfies $\Sigma \subset \mathcal{I}^+(N)$. Denote $d_N|_\Sigma$ by the Lorentzian distance function with respect to N on Σ . Then*

$$\Delta_\Sigma(d_N|_\Sigma)(q) \geq \overline{\text{Hess}}(d_N(q))(\nu, \nu) + 3H_\Sigma(q) \sqrt{1 + |\nabla(d_N|_\Sigma)(q)|^2} - 3H_N(p), \quad (8)$$

where ν and H_Σ are the future-directed unit timelike vector field and the mean curvature of Σ , respectively, H_N is the future mean curvature of N along the orthogonal projection of Σ on N , and p is the orthogonal projection of q on N .

Proof. Lemma 4 and equation (7) imply

$$\begin{aligned} \overline{\Delta} d_N(q) &= \Delta_\Sigma(d_N|_\Sigma)(q) - \overline{\text{Hess}}(d_N(q))(\nu, \nu) - 3H_\Sigma(q) \sqrt{1 + |\nabla(d_N|_\Sigma)(q)|^2} \\ &\geq -3H_N(p), \end{aligned}$$

so we have

$$\Delta_\Sigma(d_N|_\Sigma)(q) \geq \overline{\text{Hess}}(d_N(q))(\nu, \nu) + 3H_\Sigma(q) \sqrt{1 + |\nabla(d_N|_\Sigma)(q)|^2} - 3H_N(p).$$

□

Next theorem states that the longest timelike geodesic between two spacelike hypersurfaces will be perpendicular to both hypersurfaces.

Theorem 6. *Let N and Σ be two submanifolds of L such that $\Sigma \subset \mathcal{I}^+(N)$, and let $\gamma : [0, l] \rightarrow L$ be a future directed timelike geodesic such that $\gamma(0) \in N, \gamma(l) \in \Sigma$ and γ is the longest curve from N to Σ . Then $\gamma'(0)$ is perpendicular to N and $\gamma'(l)$ is perpendicular to Σ .*

Proof. Suppose that $\gamma'(0)$ is not perpendicular to N . Choose $v \in T_{\gamma(0)}N$ such that $\langle \gamma'(0), v \rangle > 0$, and let $\alpha(s)$ be a curve in N starting at $\gamma(0)$ such that $\alpha'(0) = v$. We construct a variation $\Phi : [0, t] \times (-\varepsilon, \varepsilon) \rightarrow L$ of $\gamma(t)$ such that

$$\Phi(t, 0) = \gamma(t), \quad \Phi(0, s) = \alpha(s), \quad \text{and} \quad \Phi(l, s) = \gamma(l).$$

Denote $\gamma_s(t) = \Phi(t, s)$ the family of timelike curves. Then the first variation formula (see [3, Proposition 2, page 264]) implies

$$\begin{aligned} \left. \frac{d}{ds} \text{Length}(\gamma_s) \right|_{s=0} &= \frac{1}{l} \left(-\langle \gamma'(t), v \rangle \Big|_{t=0}^{t=l} + \int_0^l \langle v, \nabla_{\gamma'(t)} \gamma'(t) \rangle dt \right) \\ &= \frac{1}{l} \langle \gamma'(0), v \rangle > 0. \end{aligned}$$

Therefore, for small s , $\text{Length}(\gamma_s) > \text{Length}(\gamma)$, which implies $\gamma(t)$ is not longest curve from N to Σ , and hence $\gamma'(0)$ must be perpendicular to L .

Similar argument will get $\gamma'(l)$ is perpendicular to N . \square

4 Dirichlet problem for SS-CMC equation

4.1 Setting the SS-CMC Dirichlet problem

Let $\Sigma : (T = F(X), X, \theta, \phi)$ be an SS-CMC hypersurface in the Kruskal extension. In [8], we have computed the SS-CMC equation:

$$\begin{aligned} F''(X) + e^{-\frac{r}{2M}} \left(\frac{6M}{r^2} - \frac{1}{r} \right) (-F(X) + F'(X)X)(1 - (F'(X))^2) \\ + \frac{12HMe^{-\frac{r}{4M}}}{\sqrt{r}} (1 - (F'(X))^2)^{\frac{3}{2}} = 0, \end{aligned} \quad (9)$$

where the spacelike condition is equivalent to $1 - (F'(X))^2 > 0$, and where H is the constant mean curvature. Remark that $r = r(T, X) = r(F(X), X)$ satisfies the equation (2), namely, $(r - 2M)e^{\frac{r}{2M}} = X^2 - T^2 = X^2 - (F(X))^2$; spherically symmetric condition means that the function $T = F(X)$ is independent of θ and ϕ , and M is the mass of the Schwarzschild spacetime, which is a positive constant.

We can formulate the SS-CMC Dirichlet problem as follows:

(\star) **Dirichlet problem for the SS-CMC equation with symmetric boundary data.** Given $H \in \mathbb{R}$ and boundary data $(T_0, X_0, \theta, \phi), (T_0, -X_0, \theta, \phi)$ in the Kruskal extension, does there exist a unique hypersurface $\Sigma : (T = F(X), X, \theta, \phi)$ satisfying the SS-CMC equation (9), the spacelike condition $1 - (F'(X))^2 > 0$, and the boundary value condition $F(X_0) = F(-X_0) = T_0$?

Since we consider SS-CMC hypersurfaces in this paper, the following discussions we only write the T - X coordinates (T, X) instead of the full coordinates (T, X, θ, ϕ) in convenience.

4.2 Existence of the SS-CMC equation

Theorem 7. *Dirichlet problem for the SS-CMC equation with symmetric boundary data (\star) is solvable.*

The idea to the proof of the existence is the shooting method. Take boundary data (T_0, X_0) and $(T_0, -X_0)$ in the middle region for example, and see Figure 5. Consider the family of SS-CMC hypersurfaces $\Sigma_{H,c}$ with $c \in (c_H, C_H)$ passing through (T_0, X_0) . The family $\Sigma_{H,c}$ is continuously varied with respect to the parameter c . When we observe the position of the “throat” $(t_{H,c}, r_{H,c})$ of each SS-CMC hypersurface $\Sigma_{H,c}$, if $c \rightarrow c_H$ or $c \rightarrow C_H$, then $r_{H,c}$ will tend to $r = r_H$ or $r = R_H$, respectively. By symmetry and the Intermediate Value Theorem, we know that there must be some $\Sigma_{H,c'}, c' \in (c_H, C_H)$ passing through the other boundary data $(T_0, -X_0)$.

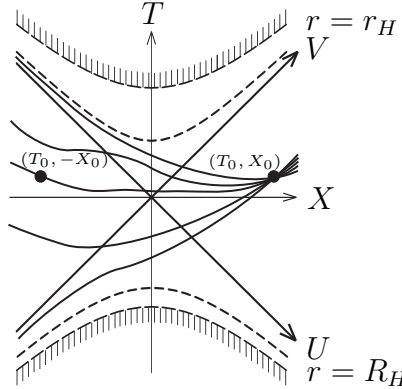


Figure 5: Existence of the SS-CMC Dirichlet problem.

Proof. (A) For (T_0, X_0) and $(T_0, -X_0)$ in the region between $r = r_H$ and $r = R_H$, denote (t_0, r_0) by the standard Schwarzschild coordinates of (T_0, X_0) . For $c \in (c_H, C_H)$, consider the following curve in the Kruskal extension in null coordinates:

$$\alpha(c) = (U(c), V(c)) = (U(t_{H,c}, r_{H,c}), V(t_{H,c}, r_{H,c})),$$

where

$$U(c) = \begin{cases} -e^{-\frac{1}{4M}(t_0-2r_{H,c}-4M \ln|r_{H,c}-2M|+r_0+2M \ln|r_0-2M|+\int_{r_0}^{r_{H,c}} \bar{f}'(x,c) dx)} & \text{in region } \Pi \\ +e^{-\frac{1}{4M}(t_0-r_0-2M \ln|r_0-2M|+\int_{r_0}^{r_{H,c}} \bar{f}'(x,c) dx)} & \text{in region } \Pi', \end{cases}$$

$$V(c) = \begin{cases} +e^{\frac{1}{4M}(t_0+r_0+2M \ln|r_0-2M|+\int_{r_0}^{r_{H,c}} \bar{f}'(x,c) dx)} & \text{in region } \Pi \\ -e^{\frac{1}{4M}(t_0+2r_{H,c}+4M \ln|r_{H,c}-2M|-r_0-2M \ln|r_0-2M|+\int_{r_0}^{r_{H,c}} \bar{f}'(x,c) dx)} & \text{in region } \Pi', \end{cases}$$

and where $\bar{f}'(x, c)$ and $\tilde{f}'(x, c)$ are (5) and (6), respectively. The curve $\alpha(c)$ indicates the position of the throat $(t_{H,c}, r_{H,c})$ of the SS-CMC hypersurface $\Sigma_{H,c}$ which passes through (T_0, X_0) in the Kruskal extension. Remark that an SS-CMC hypersurface is symmetric to its throat in the standard Schwarzschild coordinates.

We will claim that $\frac{V(c')}{U(c')} = -1$ for some $c' \in (c_H, C_H)$, or $\lim_{c \rightarrow c' = -8M^3H} \frac{V(c)}{U(c)} = -1$, which means the throat of $\Sigma_{H,c'}$ is located on the T -axis, then the symmetry implies the hypersurface $\Sigma_{H,c'}$ must pass through $(T_0, -X_0)$, so we get the existence of the Dirichlet problem (\star) .

One subtle thing is that functions \bar{f}' and \tilde{f}' have different order behaviors when $c \rightarrow -8M^3H$, so we need to take care of them by the following arguments.

If $c < -8M^3H$, we know that the curve $\alpha(c)$ lies in region Π . Denote

$$\hat{r}_{H,c} = 2M + (2M - r_{H,c}) = 4M - r_{H,c} \quad \text{and} \quad \hat{t}_{H,c} = t_0 + \int_{r_0}^{\hat{r}_{H,c}} f'(x, c) dx.$$

Relation (2) gives

$$\left. \frac{V(c)}{U(c)} \right|_{\alpha(c)} = -e^{\frac{1}{2M}t_{H,c}} = -\left(e^{\frac{1}{2M}\hat{t}_{H,c}} \right) \left(e^{\frac{1}{2M}(t_{H,c}-\hat{t}_{H,c})} \right).$$

When $c \rightarrow -8M^3H^-$, we have

$$\lim_{c \rightarrow -8M^3H^-} e^{\frac{1}{2M}\hat{t}_{H,c}} = \lim_{c \rightarrow -8M^3H^-} e^{\frac{1}{2M} \left(f(r_0) + \int_{r_0}^{\hat{r}_{H,c}} f'(x, c) dx \right)} = e^{\frac{1}{2M} \left(f(r_0) + \int_{r_0}^{2M} f'(x, -8M^3H) dx \right)}.$$

The limit exists because the function $f'(x, -8M^3H)$ is of order $O(r - 2M)^{-\frac{1}{2}}$. Next, we investigate the other limit:

$$\lim_{c \rightarrow -8M^3H^-} e^{\frac{1}{2M}(t_{H,c}-\hat{t}_{H,c})} = \lim_{c \rightarrow -8M^3H^-} e^{\frac{1}{2M} \left(2(2M-r_{H,c}) + \int_{r_{H,c}}^{4M-r_{H,c}} \bar{f}'(x, c) dx \right)}.$$

Since

$$\bar{f}'(x, c) = \frac{x^4}{(x - r_{H,c})^{\frac{1}{2}} P(x, c)},$$

where $P(x, c) > 0$ is a smooth function, let $y = x - r_{H,c}$, we get

$$\begin{aligned} \int_{r_{H,c}}^{4M-r_{H,c}} \bar{f}'(x, c) dx &= \int_0^{2(2M-r_{H,c})} \frac{(y+r_{H,c})^4}{y^{\frac{1}{2}} P(y+r_{H,c}, c)} dy \leq \int_0^{2(2M-r_{H,c})} \frac{M}{y^{\frac{1}{2}}} dy \\ &= 8M(2M-r_{H,c}) \rightarrow 0 \text{ as } c \rightarrow -8M^3H^-. \end{aligned}$$

Hence the limit

$$\lim_{c \rightarrow -8M^3H^-} \left. \frac{V(c)}{U(c)} \right|_{\alpha(c)} = -e^{\frac{1}{2M}(f(r_0) - \int_{2M}^{r_0} f'(x, -8M^3H) dx)}$$

exists. We denote the limit by L_0 . On the other hand, from the limit behavior of SS-CMC hypersurfaces in paper [8], we know

$$\lim_{c \rightarrow c_H^+} \left. \frac{V(c)}{U(c)} \right|_{\alpha(c)} = 0.$$

If $c > -8M^3H$, we know that the curve lies in region Π' , and we get

$$\frac{V(c)}{U(c)} = -e^{\frac{1}{2M}t_{H,c}} = -e^{\frac{1}{2M}(t_0 + r_{H,c} + 2M \ln |r_{H,c} - 2M| - r_0 - 2M|r_0 - 2M| + \int_{r_0}^{r_{H,c}} \bar{f}'(x, c) dx)}.$$

Similar discussion gives the result that

$$\lim_{c \rightarrow -8M^3H^+} \left. \frac{V(c)}{U(c)} \right|_{\alpha(c)} = -e^{\frac{1}{2M}(f(r_0) - \int_{2M}^{r_0} f'(x, -8M^3H) dx)} = L_0.$$

Furthermore, from the limit behavior of SS-CMC hypersurfaces in paper [8], we know

$$\lim_{c \rightarrow C_H^-} \left. \frac{V(c)}{U(c)} \right|_{\alpha(c)} = -\infty.$$

Therefore, by the Intermediate Value Theorem, we get

- (1) If $L_0 < -1$, there exists $c' \in (c_H, -8M^3H)$ such that $\frac{V(c')}{U(c')} = -1$.
- (2) If $L_0 = -1$, then $c' = -8M^3H$ satisfies $\lim_{c \rightarrow c'} \frac{V(c)}{U(c)} = -1$.
- (3) If $L_0 > -1$, there exists $c' \in (-8M^3H, C_H)$ such that $\frac{V(c')}{U(c')} = -1$.

(B) For (T_0, X_0) and $(T_0, -X_0)$ in the region between $r = r_H$ and $r = 0$, denote (t_0, r_0) by the standard Schwarzschild coordinates of (T_0, X_0) . Let $c_0 = -Hr_0^3 - r_0^{\frac{3}{2}}(2M - r_0)^{\frac{1}{2}}$. For $c \in (c_H, c_0)$, the curve

$$\begin{aligned} \alpha(c) &= (U(c), V(c)) \\ &= \left(-e^{-\frac{1}{4M}(\int_{r_0}^{r_{H,c}} f'(x, c) dx - r_0 - 2M \ln |r_0 - 2M|)}, e^{\frac{1}{4M}(\int_{r_0}^{r_{H,c}} f'(x, c) dx + r_0 + 2M \ln |r_0 - 2M|)} \right). \end{aligned}$$

will trace the position of the maximum radius of the Schwarzschild coordinates sphere of the SS-CMC hypersurface $\Sigma_{H,c}$ which passes through (T_0, X_0) . Since $\lim_{c \rightarrow c_0^-} \frac{V(c)}{U(c)} = -e^{\frac{1}{2M}t_{H,c}} < -1$ and $\lim_{c \rightarrow c_H^+} \frac{V(c)}{U(c)} = 0$, by the Intermediate Value Theorem, there exists $c' \in (c_H, c_0)$ such that $\frac{V(c')}{U(c')} = -1$.

(C) For (T_0, X_0) and $(T_0, -X_0)$ in the region between $r = R_H$ and $r = 0$, denote (t_0, r_0) by the standard Schwarzschild coordinates of (T_0, X_0) . Let $c_0 = -Hr_0^3 + r_0^{\frac{3}{2}}(2M - r_0)^{\frac{1}{2}}$. For $c \in (c_0, C_H)$, the curve

$$\begin{aligned} \alpha(c) &= (U(c), V(c)) \\ &= \left(e^{-\frac{1}{4M} \left(\int_{r_0}^{r_{H,c}} f'(x, c) dx - r_0 - 2M \ln |r_0 - 2M| \right)}, -e^{\frac{1}{4M} \left(\int_{r_0}^{r_{H,c}} f'(x, c) dx + r_0 + 2M \ln |r_0 - 2M| \right)} \right). \end{aligned}$$

will trace the position of the maximum radius of the Schwarzschild coordinates sphere in the SS-CMC hypersurface $\Sigma_{H,c}$ which passes through (T_0, X_0) . Since $\lim_{c \rightarrow c_0^+} \frac{V(c)}{U(c)} = -e^{\frac{1}{2M}t_{H,c}} > -1$ and $\lim_{c \rightarrow C_H^-} \frac{V(c)}{U(c)} = -\infty$, by the Intermediate Value Theorem, there exists $c' \in (c_0, C_H)$ such that $\frac{V(c')}{U(c')} = -1$. \square

It is easy to find that solutions of the SS-CMC equation with symmetric boundary data are symmetric about the T -axis in the Kruskal extension.

Definition 8. We say an SS-CMC hypersurface $\Sigma : (T = F(X), X, \theta, \phi)$ in the Kruskal extension is *T -axisymmetric* if $F(-X) = F(X)$ for all X .

Theorem 9. All SS-CMC hypersurfaces satisfying the Dirichlet problem (\star) are *T -axisymmetric*. That is, $F(-X) = F(X)$ for all X .

Proof. Suppose that an SS-CMC hypersurface Σ satisfying (\star) is not T -axisymmetric. We consider its T -axisymmetric reflection hypersurface, called $\tilde{\Sigma}$. Then Σ and $\tilde{\Sigma}$ have different parameter c . On the other hand, the reflection does not change the radius of the throat $r = r_{H,c}$ or the maximum radius of the Schwarzschild coordinates sphere, so Σ and $\tilde{\Sigma}$ must have the same value c , and it leads to the contradiction. \square

4.3 Uniqueness of the SS-CMC equation

We will prove the uniqueness of the SS-CMC Dirichlet problem in this subsection. Before that, we need to find more properties about the SS-CMC hypersurfaces. Theorem 7 and 9 imply that for fixed boundary data (T_0, X_0) and $(T_0, -X_0)$, and for every $H \in \mathbb{R}$, there exists a TSS-CMC hypersurface $\Sigma_{H,c}$ satisfying the Dirichlet problem (\star) . Next theorem shows that these hypersurfaces $\Sigma_{H,c}$ are continuously varied with respect to the mean curvature H .

Theorem 10. For every $H \in \mathbb{R}$, denote $\Sigma_{H,c(H)} : (T = F_H(X), X, \theta, \phi)$ an SS-CMC hypersurface satisfying the SS-CMC equation (9), the spacelike condition $1 - (F'_H(X))^2 > 0$, and the boundary value condition $F_H(X_0) = F_H(-X_0) = T_0$. Then the family $\Sigma_{H,c(H)}$ (the function $F_H(X)$) is continuously varied with respect to the mean curvature H .

Proof. (A) For (T_0, X_0) and $(T_0, -X_0)$ between $r = r_H$ and $r = R_H$, consider two functions

$$\bar{G}(r, H) = t_0 - r - 2M \ln |r - 2M| + r_0 + 2M \ln |r_0 - 2M| - \int_r^{r_0} \bar{f}'(x, H, r) dx,$$

where

$$\begin{aligned} \bar{f}'(x, H, r) &= \frac{x^4}{(Hx^3 + c)^2 + x^3(x - 2M) - (Hx^3 + c)\sqrt{(Hx^3 + c)^2 + x^3(x - 2M)}}, \\ c &= c(H, r) = -Hr^3 - r^{\frac{3}{2}}(2M - r)^{\frac{1}{2}}, \end{aligned}$$

and

$$\tilde{G}(r, H) = t_0 + r + 2M \ln |r - 2M| - r_0 - 2M \ln |r_0 - 2M| - \int_r^{r_0} \tilde{f}'(x, H, r) dx,$$

where

$$\begin{aligned} \tilde{f}'(x, H, r) &= \frac{-x^4}{(Hx^3 + c)^2 + x^3(x - 2M) + (Hx^3 + c)\sqrt{(Hx^3 + c)^2 + x^3(x - 2M)}}, \\ c &= c(H, r) = -Hr^3 + r^{\frac{3}{2}}(2M - r)^{\frac{1}{2}}. \end{aligned}$$

Both domain of \bar{G} and \tilde{G} are a subset of $(0, 2M] \times \mathbb{R}$. Since we have the relation

$$\frac{V(r, H)}{U(r, H)} = -e^{-\frac{1}{2M}\bar{G}(r, H)} \quad (\text{region } \Pi) \quad \text{and} \quad \frac{V(r, H)}{U(r, H)} = -e^{-\frac{1}{2M}\tilde{G}(r, H)} \quad (\text{region } \Pi'),$$

we use the solutions of equations $\bar{G}(r, H) = 0$ or $\tilde{G}(r, H) = 0$ to label the throat of TSS-CMC hypersurfaces satisfying the Dirichlet problem (\star) . More precisely, Theorem 7 shows that for every $H \in \mathbb{R}$, there exists $r = r(H)$ such that $\bar{G}(r, H) = 0$ or $\tilde{G}(r, H) = 0$. Hence the set (r, H) of $\bar{G}(r, H) = 0$ or $\tilde{G}(r, H) = 0$ correspond to TSS-CMC hypersurfaces $\Sigma_{H,c(H)}$.

We compute the partial derivative of the function $\bar{G}(r, H)$ with respect to H :

$$\begin{aligned} \frac{\partial \bar{G}}{\partial H} &= - \int_r^{r_0} \frac{\partial \bar{f}'}{\partial H}(x, H, r) dx = - \int_r^{r_0} \frac{\partial f'}{\partial H}(x, H, r) dx \\ &= - \int_r^{r_0} \frac{1}{h(x)} \frac{\frac{\partial l}{\partial H}(x, H, r)}{(1 + l^2(x, H, r))^{\frac{3}{2}}} dx < 0. \end{aligned}$$

By the Implicit Function Theorem, for every (r, H) with $\bar{G}(r, H) = 0$, there exists an interval $(r - \delta, r + \delta)$ such that the solution of $\bar{G}(r, H) = 0$ in the interval can be written as a graph of a function $H = H(r)$.

Similarly, we compute

$$\begin{aligned}\frac{\partial \tilde{G}}{\partial H} &= - \int_r^{r_0} \frac{\partial \tilde{f}'}{\partial H}(x, H, r) dx = - \int_r^{r_0} \frac{\partial f'}{\partial H}(x, H, r) dx \\ &= - \int_r^{r_0} \frac{1}{h(x)} \frac{\frac{\partial l}{\partial H}(x, H, r)}{(1 + l^2(x, H, r))^{\frac{3}{2}}} dx < 0.\end{aligned}$$

By the Implicit Function Theorem, for every (r, H) with $\tilde{G}(r, H) = 0$, there exists an interval $(r - \delta, r + \delta)$ such that the solution of $\tilde{G}(r, H) = 0$ in the interval can be written as a graph of a function $H = H(r)$.

(B) For (T_0, X_0) and $(T_0, -X_0)$ between $r = r_H$ and $r = 0$ (between $r = 0$ and $r = R_H$), consider the function

$$G(r, H) = t_0 + \int_{r_0}^r f'(x, c) dx \quad \text{with} \quad \frac{V(r, H)}{U(r, H)} = -e^{-\frac{1}{2M}G(r, H)}$$

in region Π (in region Π'). Since $\frac{\partial G}{\partial H} > 0$ in region Π ($\frac{\partial G}{\partial H} > 0$ in region Π'), by the Implicit Function Theorem, for every (r, H) with $G(r, H) = 0$, there exists an interval $(r - \delta, r + \delta)$ such that the solution of $G(r, H) = 0$ in the interval can be written as a graph of a function $H = H(r)$. \square

The result of Theorem 10 is illustrated in Figure 6. We plot the graph of $H = H(r)$ in the r - H plane.

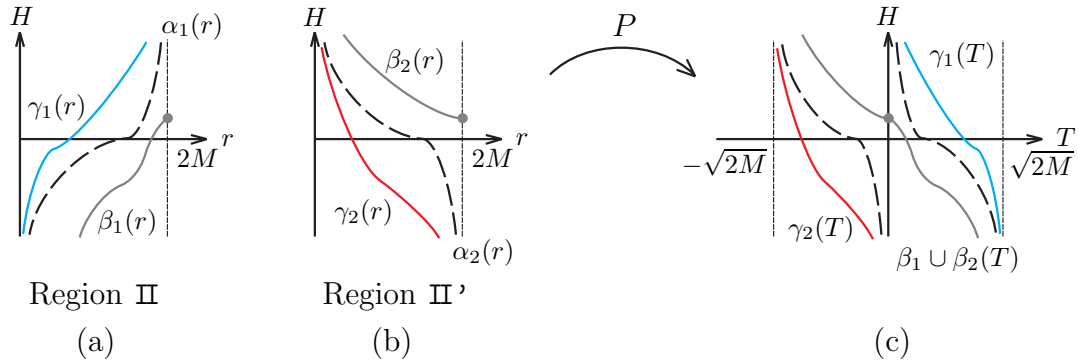


Figure 6: (a) and (b): Curves $\beta_1(r)$, $\beta_2(r)$, and $\gamma_1 \cup \gamma_2(r)$ are solutions of $\bar{G}(r, H) = 0$, $\tilde{G}(r, H) = 0$, and $G(r, H) = 0$, respectively. They correspond to TSS-CMC hyper-surfaces $\Sigma_{H, c(H)}$ for different mean curvature H . (c) We map all curves in Figure 6 (a) and (b) to the T - H plane.

In Figure 6 (a), the dotted curve $\alpha_1(r)$ is the graph of $H(r) = \frac{2r-3M}{3\sqrt{r^3(2M-r)}}$. Each point on $\alpha_1(r)$ represents a cylindrical hypersurface with constant mean curvature H

in region Π . Curves $\beta_1(r)$ and $\gamma_1(r)$ are solutions of $\bar{G}(r, H) = 0$ and $G(r, H) = 0$, respectively. Similarly, in Figure 6 (b), the dotted curve $\alpha_2(r)$ is the graph of $H(r) = \frac{3M-2r}{3\sqrt{r^3(2M-r)}}$. Curves $\beta_2(r)$ and $\gamma_2(r)$ are solutions of $\tilde{G}(r, H) = 0$ and $G(r, H) = 0$, respectively. Theorem 10 implies that every curve $\beta_1, \beta_2, \gamma_1$, and γ_2 is the graph of some continuous function $H = H(r)$. Theorem 7 implies that $\beta_1(2M) = \beta_2(2M)$.

Consider a map P from these two r - H planes in Figure 6 (a) and (b) to the T - H plane in Figure 6 (c) by

$$H \xrightarrow{P} H \quad \text{and} \quad r \xrightarrow{P} T = \begin{cases} \sqrt{2M-r} e^{\frac{r}{4M}} & \text{if } r \in (0, 2M] \text{ in region } \Pi \\ -\sqrt{2M-r} e^{\frac{r}{4M}} & \text{if } r \in (0, 2M] \text{ in region } \Pi' \end{cases}$$

Theorem 10 shows that each curve $\beta_1 \cup \beta_2(T)$, $\gamma_1(T)$, and $\gamma_2(T)$ is the graph of some continuous function $H = H(T)$ in the T - H plane.

In order to prove the uniqueness of the SS-CMC Dirichlet problem (\star) , it is equivalent to prove that each curve $\beta_1 \cup \beta_2(T)$, $\gamma_1(T)$ and $\gamma_2(T)$ is the graph of a “decreasing” function in the T - H plane.

Theorem 11. *The solution of the Dirichlet problem for SS-CMC equation with symmetric boundary data (\star) is unique.*

Proof. (A) Given $H_0 \in \mathbb{R}$, suppose that there are two TSS-CMC hypersurfaces $\Sigma_1 : (T = F_1(X), X)$ and $\Sigma_2 : (T = F_2(X), X)$ with constant mean curvature H_0 satisfying the Dirichlet problem (\star) , and suppose that $F_1(X) < F_2(X)$ in $(-X_0, X_0)$. Then the curve $\beta_1 \cup \beta_2(T)$ passes through $(T_1 = F_1(X=0), H_0)$ and $(T_2 = F_2(X=0), H_0)$ in the T - H plane, and $\beta_1 \cup \beta_2(T)$ is no longer the graph of a decreasing function in the T - H plane, which implies that there must be some increasing part on some interval $I \subset [T_1, T_2]$. See Figure 7. Then we can take two TSS-CMC hypersurfaces N and Σ such that $\Sigma \subset \mathcal{I}^+(N)$ in the region $T \times [-X_0, X_0]$ in the Kruskal extension and $H_N < H_\Sigma$.

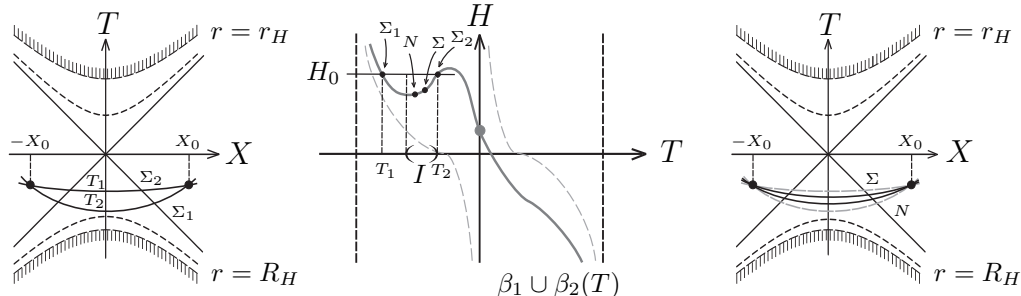


Figure 7: Choose two hypersurfaces N and Σ in the increasing part of $\beta_1 \cup \beta_2(T)$, and then find a contradiction to the nonuniqueness of the SS-CMC Dirichlet problem.

Consider $d_N|_\Sigma$ the Lorentzian distance function restricted on the spacelike hypersurface Σ . Since $N \neq \Sigma$, we know that the maximum value of $d_N|_\Sigma$ is positive and will achieve at some point q , which is in the interior of $\Sigma \cap T \times (-X_0, X_0)$ in the Kruskal extension. We apply the estimate (8) to N and Σ and get

$$\begin{aligned} 0 \geq \Delta_\Sigma(d_N|_\Sigma)(q) &\geq \overline{\text{Hess}}(d_N(q))(\nu, \nu) + 3H_\Sigma(q)\sqrt{1 + |\nabla(d_N|_\Sigma)(q)|^2} - 3H_N(p) \\ &\geq 0 + 3H_\Sigma(q) - 3H_N(p) > 0, \end{aligned}$$

which is a contradiction. Remark that $\overline{\text{Hess}}(d_N(q))(\nu, \nu) = 0$ because of the perpendicular property in Theorem 6. \square

5 Applications to CMC foliation conjecture

Malec and Ó Murchadha in [11] constructed a family of T -axisymmetric SS-CMC (TSS-CMC) hypersurfaces in the Kruskal extension. Each TSS-CMC hypersurface in this family shares the same constant mean curvature H . They conjectured this TSS-CMC family will foliate the Kruskal extension in [11] without rigorous mathematical proof.

From the viewpoint of the SS-CMC Dirichlet problem, we will prove the existence of the TSS-CMC foliation in the Kruskal extension. Given $H \in \mathbb{R}$, for all symmetric pairs (T_0, X_0) and $(T_0, -X_0)$ in the Kruskal extension, we collect all TSS-CMC hypersurfaces with mean curvature H and pass through (T_0, X_0) and $(T_0, -X_0)$. We denote this TSS-CMC family by $\{\Sigma_H\}$. Since (T_0, X_0) and $(T_0, -X_0)$ will be taken in the whole Kruskal extension, $\{\Sigma_H\}$ must cover the whole Kruskal extension. If any two hypersurfaces $\Sigma_1, \Sigma_2 \in \{\Sigma_H\}$ are not disjoint, then they must intersect at some symmetric pairs. However, the uniqueness of the SS-CMC Dirichlet problem implies $\Sigma_1 = \Sigma_2$. Therefore, we can conclude the following theorem:

Theorem 12. *The existence and uniqueness of the Dirichlet problem (\star) is equivalent to the existence of TSS-CMC foliation in the Kruskal extension.*

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